

Approximate counting of regular hypergraphs

Andrzej Dudek^{*} Alan Frieze[†] Andrzej Ruciński[‡] Matas Šileikis[§]

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Abstract

In this paper we approximately count d -regular k -uniform hypergraphs on n vertices, provided k is fixed and $d = d(n) = o(n^\kappa)$, where $\kappa = \kappa(k) = 1$ for all $k \geq 4$, while $\kappa(3) = \frac{1}{2}$. In doing so, we extend to hypergraphs a switching technique of McKay and Wormald.

1 Introduction

We consider k -uniform hypergraphs (or k -graphs, for short) on the vertex set $V = [n] := \{1, \dots, n\}$. A k -graph $H = (V, E)$ is d -regular, if the degree of every vertex $v \in V$, $\deg_H(v) := \deg(v) := |\{e \in E : v \in e\}|$ equals d .

Let $\mathcal{H}^{(k)}(n, d)$ be the class of all d -regular k -graphs on $[n]$. Note that each $H \in \mathcal{H}^{(k)}(n, d)$ has $m := nd/k$ edges (throughout, we implicitly assume that $k|nd$). We treat d as a function of n , possibly constant.

^{*}Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA. Research supported in part by Simons Foundation Grant #244712.

[†]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA. Research supported in part by NSF Grant CCF2013110.

[‡]Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland. Research supported by the Polish NSC grant N201 604940 and the NSF grant DMS-1102086. Part of research performed at Emory University, Atlanta.

[§]Department of Mathematics, Uppsala University, Sweden. Research supported by the Polish NSC grant N201 604940. Part of research performed at Adam Mickiewicz University, Poznań.

A result of McKay [6] contains an asymptotic formula for the number of n -vertex d -regular graphs, when $d \leq \varepsilon n$ for any constant $\varepsilon < 2/9$. In this paper we present an asymptotic enumeration of all d -regular k -graphs on a given set of n vertices, where $k \geq 3$ and $d = d(n)$ is either a constant or does not grow with n too quickly. Let $\kappa = \kappa(k) = 1$ for $k \geq 4$ and $\kappa(3) = \frac{1}{2}$.

Theorem 1. *For every $k \geq 3$ and $1 \leq d = o(n^\kappa)$,*

$$|\mathcal{H}^{(k)}(n, d)| = \frac{(nd)!}{(nd/k)!(k!)^{nd/k}(d!)^n} \exp \left\{ -\frac{1}{2}(k-1)(d-1)(1 + O(\delta(n))) \right\},$$

where $\delta(n) = (dn)^{-1/2} + d/n$.

Theorem 1 extends a result from [4] where Cooper, Frieze, Molloy and Reed proved that formula for d fixed using the by now standard *configuration model* (see [1, 2, 9] for the graph case). Already for graphs, in [6], and later in [7] and [8], this technique was combined with the idea of *switchings*, a sequence of operations on a graph which eliminate loops and multiple edges, while keeping the degrees unchanged and leading to an *almost* uniform distribution of the simple graphs obtained as the ultimate outcome (but see Remark 3 in Section 3).

To prove Theorem 1 we apply these ideas together with a modification from [3], where instead of configurations, permutations were used to generate graphs with a given degree sequence. To describe this modification, consider a generalization of a k -graph in which edges are multisets of vertices rather than just sets. By a *k -multigraph* we mean a pair $H = (V, E)$ where V is a set and E is a multiset of k -element multisubsets of V . Thus we allow both multiple edges and loops, a *loop* being an edge which contains more than one copy of a vertex. We call an edge *proper* if it is not a loop. We say that a k -multigraph is *simple* if it is a k -graph, that is, if it contains neither multiple edges nor loops. Henceforth, for brevity of notation, we denote an edge of a k -graph by $v_1 \dots v_k$ rather than $\{v_1, \dots, v_k\}$.

Given a sequence $\mathbf{x} \in [n]^{ks}$, $s \in \mathbb{N}$, let $H(\mathbf{x})$ stand for the k -multigraph with edge multiset $E = \{x_{ki+1}, \dots, x_{ki+k} : i = 0, \dots, s-1\}$ and let $\lambda(\mathbf{x})$ be the number of loops in

$H(\mathbf{x})$.

Let $\mathcal{P} = \mathcal{P}(n, d)$ be the family of all permutations of the multiset

$$\left(\underbrace{1, \dots, 1}_d, \underbrace{2, \dots, 2}_d, \dots, \underbrace{n, \dots, n}_d \right).$$

Note that $|\mathcal{P}| = (nd)!(d!)^{-n}$. Let $\mathbf{Y} = \mathbf{Y}_{n,d} = (Y_1, \dots, Y_{nd})$ be a sequence chosen uniformly at random from \mathcal{P} .

In the next section we sketch a proof of Theorem 1 together with some auxiliary results. More details of these proofs can be found in appendix A.

2 Proof of Theorem 1

2.1 Setup

Let \mathcal{E} be the family of those permutations $\mathbf{y} \in \mathcal{P}$ for which the k -multigraph $H(\mathbf{y})$ has no multiple edges and contains at most

$$L := \sqrt{dn}$$

loops, but no loops of other type than $x_1 x_1 x_2 \dots x_{k-1}$, where x_1, \dots, x_{k-1} are distinct vertices. Let

$$\mathcal{E}_l = \{\mathbf{y} \in \mathcal{E} : \lambda(\mathbf{y}) = l\}, \quad l = 0, \dots, L.$$

Note that

$$\mathcal{E}_0 = \left\{ \mathbf{y} \in \mathcal{P} : H(\mathbf{y}) \in \mathcal{H}^{(k)}(n, d) \right\}$$

is precisely the family of those permutations from \mathcal{P} which represent simple k -graphs. In turn, for each $H \in \mathcal{H}^{(k)}(n, d)$ there are $(nd/k)!(k!)^{nd/k}$ permutations $\mathbf{y} \in \mathcal{E}_0$ with $H(\mathbf{y}) = H$.

Therefore, in order to prove Theorem 1, it suffices to show that

$$|\mathcal{P}|/|\mathcal{E}_0| = \exp \left\{ \frac{1}{2}(k-1)(d-1)(1 + O(\delta(n))) \right\}. \quad (1)$$

Our plan is as follows. First, in Proposition 2, we will prove that

$$|\mathcal{P}| = (1 + O(\phi_k(n)))|\mathcal{E}|, \quad (2)$$

where $\phi_k(n) = \sqrt{d/n}$ for $k \geq 4$, while $\phi_3(n) = d\delta(n)$. (Note that $\phi_k(n) \leq d\delta(n)$ for every $k \geq 3$.) Then, we will show (1) with $|\mathcal{E}|$ in place of $|\mathcal{P}|$, by writing

$$\frac{|\mathcal{E}|}{|\mathcal{E}_0|} = \sum_{l=0}^L \frac{|\mathcal{E}_l|}{|\mathcal{E}_0|},$$

and estimating the ratio $|\mathcal{E}_l|/|\mathcal{E}_{l-1}|$ uniformly for every $1 \leq l \leq L$ (see Claim 4).

In what follows it will be convenient to work directly with permutation \mathbf{Y} rather than with the k -multigraph $H(\mathbf{Y})$ generated by it. Recycling the notation, we still call consecutive k -tuples $(Y_{ki+1}, \dots, Y_{ki+k})$ of \mathbf{Y} *edges*, *proper edges*, or *loops*, whatever appropriate. E.g., we say that \mathbf{Y} contains *multiple edges*, if $H(\mathbf{Y})$ contains multiple edges, that is, some two edges of \mathbf{Y} are identical as multisets. We use the standard notation $(x)_a = x(x-1) \cdots (x-a+1)$.

The following proposition implies (2), because $\mathbb{P}(\mathbf{Y} \in \mathcal{E}) = |\mathcal{E}|/|\mathcal{P}|$.

Proposition 2. *If $k \geq 3$ and $1 \leq d = o(n^\kappa)$, then $\mathbb{P}(\mathbf{Y} \in \mathcal{E}) = 1 - O(\phi_k(n))$.*

A simple proof of Proposition 2 is based on the first moment method. In particular, the expectations of the number of pairs of multiple edges, edges with a vertex of multiplicity at least 3 or two vertices of multiplicity 2 and edges which are loops are, respectively, $O(d^2 n^{2-k}) = O((d/n^\kappa)^2)$, $O(d/n)$, and $\mathbb{E}\lambda(\mathbf{Y}) \sim \frac{k-1}{2}(d-1)$. The last formula implies that $\mathbb{P}(\lambda(\mathbf{Y}) > L) \leq \frac{\mathbb{E}\lambda(\mathbf{Y})}{L} = O(d^{1/2} n^{-1/2})$.

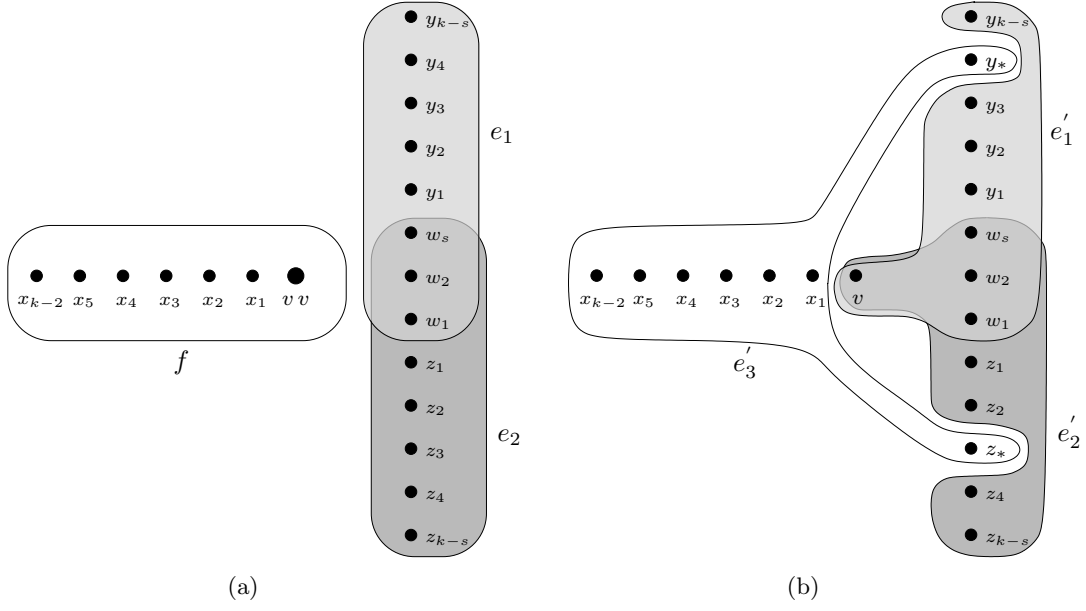


Figure 1: Switching (a) before and (b) after.

2.2 Switchings

Now we define an operation, called *switching*, which generalizes to k -graphs a graph switching introduced in [7] (see also [8]). Permutations $\mathbf{y} \in \mathcal{E}_l$, $\mathbf{z} \in \mathcal{E}_{l-1}$ are said to be *switchable*, if \mathbf{z} can be obtained from \mathbf{y} by the following operation. From the edges of \mathbf{y} , choose a loop f and two proper edges e_1, e_2 that are disjoint from f and share at most $k-2$ vertices (see Figure 1(a)). Letting $s = |e_1 \cap e_2|$, write

$$f = vx_1 \dots x_{k-2}, \quad e_1 = w_1 \dots w_s y_1 \dots y_{k-s}, \quad e_2 = w_1 \dots w_s z_1 \dots z_{k-s}.$$

Select vertices $y_* \in \{y_1, \dots, y_{k-s}\}$ and $z_* \in \{z_1, \dots, z_{k-s}\}$, and replace f, e_1 , and e_2 by three proper edges

$$e'_1 = e_1 \cup \{v\} - \{y_*\}, \quad e'_2 = e_2 \cup \{v\} - \{z_*\}, \quad e'_3 = f \cup \{y_*, z_*\} - \{v, v\}$$

as in Figure 1(b). Since we are dealing with permutations, for definiteness let us assume that the procedure is performed by swapping with y_* the copy of v which appears in \mathbf{y}

further to the left and with z_* the one further to the right.

Note that given f, e_1 , and e_2 as above, some choice of y_* and z_* might not yield a permutation $\mathbf{z} \in \mathcal{E}_{l-1}$, because one or more of e'_i 's might already be present in \mathbf{y} . We denote the number of admissible choices of (y_*, z_*) by $\vec{a}(\mathbf{y}; f, e_1, e_2)$.

We can reconstruct permutations in \mathcal{E}_{l+1} which are switchable with \mathbf{y} as follows. Pick three proper edges e'_1, e'_2 , and e'_3 , the first two of which intersect and are disjoint from the third (consult with Figure 1 again). Choose a vertex v from $e'_1 \cap e'_2$ and a pair $\{y_*, z_*\}$ of vertices from e'_3 ; replace $e'_i, i = 1, 2, 3$, by a loop and two edges defined as

$$f = e'_3 \cup \{v, v\} \setminus \{y_*, z_*\}, \quad e_1 = e'_1 \cup \{y_*\} \setminus \{v\}, \quad e_2 = e'_2 \cup \{z_*\} \setminus \{v\}.$$

Again, some choices of v might not yield a permutation in \mathcal{E}_{l+1} , due to creation of multiple edges. We denote the number of admissible choices of v by $\overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3)$.

Given $\mathbf{y} \in \mathcal{E}_l$, let $F(\mathbf{y})$ and $B(\mathbf{y})$ stand, respectively, for the number of ways to perform the forward and backward switching, or, in other words, the number of permutations $\mathbf{x} \in \mathcal{E}_{l-1}$ and $\mathbf{z} \in \mathcal{E}_{l+1}$ which are switchable with \mathbf{y} . Recall that $L = \sqrt{dn}$ and set $F_l = d^2 n^2 l$, $l = 1, \dots, L$, and $B = \frac{k-1}{2} n^2 d^2 (d-1)$.

Proposition 3. *There is a sequence $\delta_1 = \delta_1(n) = O((L + d^2)/dn)$ such that for all $\mathbf{y} \in \mathcal{E}_l$, $0 < l \leq L$*

$$(1 - \delta_1)F_l \leq F(\mathbf{y}) \leq F_l \quad \text{and} \quad (1 - \delta_1)B \leq B(\mathbf{y}) \leq B.$$

Proof. We have $F(\mathbf{y}) = \sum_{\text{loop } f} \sum_{e_1, e_2} \vec{a}(\mathbf{y}; f, e_1, e_2) \leq lm^2 k^2 = n^2 d^2 l$. We say that two edges e', e'' of a k -graph are *distant* from each other if their distance in the intersection graph of $H(\mathbf{y})$ is at least three. Note that $\vec{a}(\mathbf{y}; f, e_1, e_2) = k^2$ if $e_1 \cap e_2 = \emptyset$ and both e_1 and e_2 are distant from f . Thus,

$$F(\mathbf{y}) \geq k^2(m - L - 2k^2 d^2)^2 l = k^2 m^2 l (1 - O((L + d^2)/m)).$$

To bound $B(\mathbf{y})$, note that

$$B(\mathbf{y}) = \sum_{e'_1, e'_2} \sum_{e'_3} \overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3) \leq \sum_{s=1}^{k-1} p_s(\mathbf{y}) m s \binom{k}{2} = \binom{k}{2} m \sum_{s=1}^{k-1} p_s(\mathbf{y}) s, \quad (3)$$

where $p_s(\mathbf{y})$ is the number of ordered pairs (e'_1, e'_2) of proper edges of \mathbf{y} such that $|e'_1 \cap e'_2| = s$. Further, observe that $\overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3) = s \binom{k}{2}$, whenever $|e'_1 \cap e'_2| = s$ and e'_3 is distant to both e'_1 and e'_2 . We also have

$$\sum_{s=1}^{k-1} p_s(\mathbf{y}) s = \sum_v (\deg'_v(v))_2 \leq n(d)_2, \quad (4)$$

where $\deg'_v(v)$ is the number of proper edges containing a vertex v . Thus, (3) and (4) yield an upper bound on $B(\mathbf{y})$. To bound $B(\mathbf{y})$ from below apply convexity of $f(x) = (x)_2$. \square

Claim 4. *There is a sequence $\delta_2 = \delta_2(n) = O((L + d^2)/dn)$ such that*

$$\frac{|\mathcal{E}_l|}{|\mathcal{E}_{l-1}|} = (1 \pm \delta_2) \frac{(k-1)(d-1)}{2l}$$

uniformly in $l = 1, \dots, L$.

Proof. Note that $B/F_l = \frac{(k-1)(d-1)}{2l}$. From the obvious identity

$$\sum_{\mathbf{y} \in \mathcal{E}_l} F(\mathbf{y}) = \sum_{\mathbf{z} \in \mathcal{E}_{l-1}} B(\mathbf{z})$$

and Proposition 3 we conclude that

$$\frac{(1 - \delta_1)B}{F_l} \leq \frac{|\mathcal{E}_l|}{|\mathcal{E}_{l-1}|} \leq \frac{B}{(1 - \delta_1)F_l}$$

which completes the proof. \square

It follows from Claim 4 that

$$\frac{|\mathcal{E}|}{|\mathcal{E}_0|} = \exp \left\{ \frac{1}{2}(k-1)(d-1)(1 + O(\delta_2(n))) \right\}$$

which together with (2) implies (1), hereby completing the proof of Theorem 1.

3 Concluding remarks

Remark 1. We believe that for $k = 3$ the constraint $d = o(n^{1/2})$ in Theorem 1 can be relaxed to $d = o(n)$ by allowing $O(d^2/n)$ multiple edges in $\mathbf{y} \in \mathcal{E}$ and applying an appropriate switching technique to eliminate them along with the loops.

Remark 2. In a forthcoming paper [5] we apply the switching technique presented here to embed asymptotically almost surely (*a.a.s.*) an ordinary Erdős-Rényi random k -graph $\mathbb{H}^{(k)}(n, m')$, $k \geq 3$, into a random d -regular k -graph $\mathbb{H}^{(k)}(n, d)$ for $d = \Omega(\log n)$, $d = o(\sqrt{n})$ and $m' = cnd/k$, for some constant $c > 0$. Consequently, *a.a.s.* $\mathbb{H}^{(k)}(n, d)$ inherits from $\mathbb{H}^{(k)}(n, m')$ all increasing properties held by the former model.

Remark 3. An algorithm of McKay and Wormald [7] can be easily adapted to k -graphs, yielding an expected polynomial time uniform generation of d -regular k -graphs in $\mathcal{H}^{(k)}(n, d)$. The algorithm keeps selecting a random permutation $\mathbf{y} \in \mathcal{P}$ until $\mathbf{y} \in \mathcal{E}$. Then, iteratively, a random switching is applied $\lambda(\mathbf{y})$ times to eliminate all loops and finally yield a random element of \mathcal{E}_0 . This leads to an *almost* uniform distribution over $\mathcal{H}^{(k)}(n, d)$. To make it *exactly* uniform, McKay and Wormald applied an ingenious trick of restarting the whole algorithm after every iteration of switching, say from $\mathbf{y} \in \mathcal{E}_l$ to $\mathbf{z} \in \mathcal{E}_{l-1}$, with probability $1 - (F(\mathbf{y})(1 - \delta_1)B)/(B(\mathbf{z})F_l) \leq 2\delta_1$. However, the assumption on d has to be strengthened, so that the reciprocal of the probability of not restarting the algorithm before its successful termination, or $(1 - \phi_k(n))^{-1}(1 - 2\delta_1(n))^{-L} = e^{O(\delta_1(n)L)}$, is at most a polynomial function of n . With our choice of L this imposes the bound $d = O(n^{1/3}(\log n)^{2/3})$. We may push it up to $d = O(\sqrt{n \log n})$ by redefining $L = kd + \omega(n)$ for any (sufficiently slow) sequence $\omega(n) \rightarrow \infty$. This change requires that in the last part of the proof of Proposition 2, instead

of the first moment, Chebyshev's inequality is used.

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A Appendix

Proof of Proposition 2. We will show that each of the following four statements holds with probability at least $1 - O(\phi_k(n))$:

- (i) \mathbf{Y} has no multiple edges,
 - (ii) \mathbf{Y} has no edge with a vertex of multiplicity at least 3,
 - (iii) \mathbf{Y} has no edge with two vertices of multiplicity at least 2,
 - (iv) $\lambda(\mathbf{Y}) \leq L$.
- (i) The probability that two particular edges of \mathbf{Y} are identical as multisets equals

$$\sum_{k_1 + \dots + k_n = k} \binom{k}{k_1, \dots, k_n}^2 \frac{\binom{dn-2k}{d-2k_1, \dots, d-2k_n}}{\binom{dn}{d, \dots, d}} \leq k!^2 \sum \frac{d^{2k}}{(dn)_{2k}} = O\left(n^k \frac{d^{2k}}{(dn)_{2k}}\right) = O(n^{-k}),$$

therefore, by our assumption on d , the expected number of pairs of multiple edges does not exceed

$$O\left(\binom{m}{2} n^{-k}\right) = O(d^2 n^{2-k}) = O((d/n^\kappa)^2).$$

- (ii) The expected number of edges of \mathbf{Y} having a vertex of multiplicity at least 3 is at most

$$m \times \binom{k}{3} \times n \times \frac{\binom{dn-3}{d-3, d, \dots, d}}{\binom{dn}{d, \dots, d}} = m \binom{k}{3} n \frac{(d)_3}{(dn)_3} = O(d/n).$$

- (iii) Similarly, the expected number of edges of \mathbf{Y} having at least two vertices of multiplicity at least 2 is at most

$$m \times k^4 \times n^2 \times \frac{\binom{dn-4}{d-2, d-2, d, \dots, d}}{\binom{dn}{d, \dots, d}} = m k^4 n^2 \frac{(d)_2^2}{(dn)_4} = O(d/n).$$

- (iv) In view of (ii) and (iii), it is enough to show that the number of loops of the form $x_1 x_1 x_2 x_3 \dots x_{k-1}$ does not exceed L . For $i = 1, \dots, m$, let \mathbb{I}_i be the indicator of the event

that the i 'th edge of \mathbf{Y} is such a loop. Hence, $\lambda(\mathbf{Y}) = \sum_{i=1}^m \mathbb{I}_i$. For every i we have

$$\mathbb{E} \mathbb{I}_i = \frac{\binom{k}{2}(n)_{k-1}(d)_2 d^{k-2}}{(nd)_k} \sim \binom{k}{2} \frac{d-1}{d} n^{-1}.$$

Therefore

$$\mathbb{E} \lambda(\mathbf{Y}) \sim \frac{k-1}{2} (d-1),$$

and by Markov's inequality,

$$\mathbb{P}(\lambda(\mathbf{Y}) > L) \leq \frac{\mathbb{E} \lambda(\mathbf{Y})}{L} = O(d^{1/2} n^{-1/2})$$

□

Full proof of Proposition 3. For $\mathbf{y} \in \mathcal{E}_l$, $0 < l \leq L$, we have

$$F(\mathbf{y}) = \sum_{\text{loop } f} \sum_{e_1, e_2} \vec{a}(\mathbf{y}; f, e_1, e_2), \quad (5)$$

where the inner sum is taken over ordered pairs (e_1, e_2) of proper edges disjoint from the loop f and such that $|e_1 \cap e_2| \leq k-2$. Recall that $m = nd/k$ is the number of edges in \mathbf{y} .

The number of terms in (5) is at most lm^2 and $\vec{a}(\mathbf{y}; f, e_1, e_2) \leq k^2$, therefore

$$F(\mathbf{y}) \leq lm^2 k^2 = n^2 d^2 l.$$

We say that two edges e', e'' of a k -graph are *distant* from each other if their distance in the intersection graph of H is at least three, that is, no other edge intersects both of them.

In order to bound $F(\mathbf{y})$ from below, we note that $a(\mathbf{y}; f, e_1, e_2) = k^2$ if $e_1 \cap e_2 = \emptyset$ and both e_1 and e_2 are distant from f . Indeed, then no choice of y_* or z_* leads to a multiple edge. Fix an arbitrary loop f . The number of edges *not* distant from f is then at most $k^2 d^2$. Consequently there are at least $m - l - k^2 d^2$ ways to choose e_1 . Further, given e_1 , there are at most kd edges that are distant from f but intersect e_1 , so we can choose e_2

disjoint from e_1 in at least $m - l - k^2 d^2 - kd$ ways. Since $l \leq L$ and $kd \geq 1$,

$$F(\mathbf{y}) \geq k^2(m - L - 2k^2 d^2)^2 l = k^2 m^2 l(1 + O((L + d^2)/m)).$$

Let us now proceed to the bounds for $B(\mathbf{y})$. Fix $\mathbf{y} \in \mathcal{E}_l$, $0 \leq l < L$. Then

$$B(\mathbf{y}) = \sum_{e'_1, e'_2} \sum_{e'_3} \overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3).$$

Given edges e'_1 and e'_2 such that $|e'_1 \cap e'_2| = s$, the number of choices for e'_3 is at most m , and $\overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3) \leq s \binom{k}{2}$. Therefore

$$B(\mathbf{y}) \leq \sum_{s=1}^{k-1} p_s(\mathbf{y}) m s \binom{k}{2} = \binom{k}{2} m \sum_{s=1}^{k-1} p_s(\mathbf{y}) s, \quad (6)$$

where $p_s(\mathbf{y})$ is the number of ordered pairs (e'_1, e'_2) of proper edges of \mathbf{y} such that $|e'_1 \cap e'_2| = s$. For the lower bound we observe that $\overleftarrow{a}(\mathbf{y}; e'_1, e'_2, e'_3) = s \binom{k}{2}$, whenever $|e'_1 \cap e'_2| = s$ and e'_3 is distant to both e'_1 and e'_2 . Since there are at most $2k^2 d^2$ edges not distant to e'_1 or e'_2 , we get

$$\begin{aligned} B(\mathbf{y}) &\geq \sum_{s=1}^{k-1} p_s(\mathbf{y}) (m - l - 2k^2 d^2) s \binom{k}{2} \\ &= \binom{k}{2} (m - l - 2k^2 d^2) \sum_{s=1}^{k-1} p_s(\mathbf{y}) s = \binom{k}{2} m (1 + O((L + d^2)/m)) \sum_{s=1}^{k-1} p_s(\mathbf{y}) s. \end{aligned} \quad (7)$$

Counting triples $\{(v, e'_1, e'_2) : v \in [n], v \in e'_1 \cap e'_2, e'_1 \neq e'_2\}$ in two ways we get

$$\sum_{s=1}^{k-1} p_s(\mathbf{y}) s = \sum_v (\deg'_{\mathbf{y}}(v))_2, \quad (8)$$

where $\deg'_{\mathbf{y}}(v)$ is the number of proper edges containing a vertex v . Since $x \mapsto (x)_2$ is

convex as a function defined on integers, it follows by Jensen's inequality that

$$\frac{1}{n} \sum_v (\deg'_y(v))_2 \geq \left(\frac{1}{n} \sum_v \deg'_y(v) \right)_2. \quad (9)$$

On the other hand, $\sum_v \deg'_y(v)$ is k times the number of proper edges, which is $k(m-l) = nd - kl$, therefore (9) implies

$$\sum_v (\deg'_y(v))_2 \geq n(d - kl/n)_2 = n(d)_2 (1 - O(L/m)).$$

Obviously, the L-H-S of (8) is at most $n(d)_2$, therefore by (6), (7), and (8), there exists a sequence $\delta_1 = \delta_1(n)O((L + d^2)/m)$ such that $B(y)$ satisfies the desired inequalities. \square

Detailed proof of Theorem 1. By Claim 4, there is a sequence $\delta_2 = \delta_2(n) = O((L+d^2)/dn) = O((dn)^{-1/2} + d/n)$ such that for every $l = 0, \dots, L$

$$\frac{|\mathcal{E}_l|}{|\mathcal{E}_0|} = \prod_{i=1}^l \frac{|\mathcal{E}_i|}{|\mathcal{E}_{i-1}|} = \prod_{i=1}^l \left((1 \pm \delta_2) \frac{(k-1)(d-1)}{2i} \right) = \frac{[(1 \pm \delta_2)(k-1)(d-1)/2]^l}{l!}.$$

Since $|\mathcal{E}| = \sum_{l=0}^L |\mathcal{E}_l|$, we get

$$\sum_{l=0}^L \frac{x^l}{l!} \leq \frac{|\mathcal{E}|}{|\mathcal{E}_0|} \leq \sum_{l=0}^L \frac{y^l}{l!} \quad (10)$$

where $x = (1 - \delta_2)(k-1)(d-1)/2$ and $y = (1 + \delta_2)(k-1)(d-1)/2$. As $y \geq 0$, the right inequality in (10) implies

$$\frac{|\mathcal{E}|}{|\mathcal{E}_0|} \leq \sum_{l=0}^{\infty} \frac{y^l}{l!} = e^y. \quad (11)$$

On the other hand, since $x \geq 0$, by Taylor's expansion

$$e^x \leq \sum_{l=0}^L \frac{x^l}{l!} + \frac{x^L}{L!} e^x. \quad (12)$$

Noting that $x = O(d)$ and $L! > (L/e)^L$, we have

$$\frac{x^L}{L!} = (O(d/n))^{L/2}.$$

Therefore, it follows from (10) and (12) that

$$\frac{|\mathcal{E}|}{|\mathcal{E}_0|} \geq e^x \left(1 - \frac{x^L}{L!}\right) = \exp\{x + (O(d/n))^{L/2}\}. \quad (13)$$

It follows from (11) and (13) that

$$\frac{|\mathcal{E}|}{|\mathcal{E}_0|} = \exp \left\{ \frac{1}{2}(k-1)(d-1)(1 + O(\delta_2(n))) \right\},$$

which together with (2) implies (1), hereby completing the proof. \square

Proof of $\mathbb{P}(\lambda(\mathbf{Y}) > kd + \omega(n)) = o(1)$. Let $L := kd + \omega(n)$. We will show that $\text{Var } \lambda(\mathbf{Y}) = O(d)$, from which the desired fact follows by Chebyshev's inequality:

$$\mathbb{P}(\lambda(\mathbf{Y}) > L) \leq \frac{\text{Var } \lambda(\mathbf{Y})}{(L - \mathbb{E}\lambda(\mathbf{Y}))^2} = O\left(\frac{d}{(d + \omega(n))^2}\right) = O((d + \omega(n))^{-1}) = o(1).$$

Recall that \mathbb{I}_i is the indicator that the i 'th edge of \mathbf{Y} is a loop with only one repetition, $\lambda(\mathbf{Y}) = \sum_{i=1}^m \mathbb{I}_i$, and for every i we have $\mathbb{E} \mathbb{I}_i \sim \binom{k}{2} \frac{d-1}{d} n^{-1}$. If $i \neq j$, then

$$\mathbb{E} \mathbb{I}_i \mathbb{I}_j \leq \frac{\binom{k}{2}^2 (n)_{k-1}^2 (d)_2^2 d^{2k-4}}{(nd)_{2k}},$$

therefore

$$\begin{aligned} \text{Cov}(\mathbb{I}_i, \mathbb{I}_j) &= \mathbb{E} \mathbb{I}_i \mathbb{I}_j - \mathbb{E} \mathbb{I}_i \mathbb{E} \mathbb{I}_j \\ &\leq \frac{\binom{k}{2}^2 (n)_{k-1}^2 (d)_2^2 d^{2k-4}}{(nd)_{2k}(nd)_k} ((nd)_k - (nd - k)_k) = O(n^{-3}d^{-1}). \end{aligned}$$

Finally we get

$$\text{Var } \lambda(\mathbf{Y}) = \sum_{1 \leq i \leq m} \text{Var } \mathbb{I}_i + \sum_{1 \leq i \neq j \leq m} \text{Cov}(\mathbb{I}_i, \mathbb{I}_j) = O(mn^{-1} + m^2n^{-3}d^{-1}) = O(d).$$

□